TK Solver Case Study – Order of Equations / Choice of Guess Variables

Water is draining out of the bottom of a tank through a galvanized pipe of length 2 meters and diameter 0.0158 meters. The water in the tank is 1 meter deep. Find the velocity in the pipe. The following equations can be used to solve the problem.

$$H + L = \frac{V^2}{2 \cdot g} + hf$$

$$hf = \frac{f \cdot L \cdot V^2}{2 \cdot g \cdot D}$$

$$f = \frac{0.25}{\left[\log\left[\frac{ks}{3.7 \cdot D} + \frac{5.74}{Re^{0.9}}\right]\right]^2}$$

$$Re = \frac{D \cdot V}{nu}$$

The first equation covers the energy involved from the water surface to the exit plane.

The second equation is the Darcy-Weisbach equation which relates the head loss due to friction along a given length of pipe to the average velocity of the fluid flow.

The third equation is the Swamee-Jain equation for the friction factor.

The fourth equation uses the pipe diameter, fluid velocity and fluid viscosity to compute a Reynolds number -- a dimensionless number that gives a measure of the ratio of inertial forces to viscous forces for given flow conditions. The Reynolds number is an important parameter that describes whether flow conditions lead to laminar or turbulent flow. If Re>2300 the flow is turbulent.

Here is a list of the variables with inputs and units.

Status	Input	Name	Output	Unit	Comment
	.00000114	nu		m^2/s	Kinematic Viscosity
	9.81	g		m/s^2	Acceleration due to gravity
	.00015	ks		m	Equivalent sand roughess height
	1	H		m	Height of fluid in tank
	2	L		m	Length of pipe
	.0158	D		m	Pipe diameter
		hf		m	head loss
		f			friction factor
		V		m/s	mean velocity
		Re			Reynolds number

The water viscosity and pipe roughness values are input along with the gravity value.

There are four equations and four unknowns. All the equations have two or more unknowns, so TK's Direct Solver can make no progress and an initial guess is required. The purpose of this case study is to learn why TK responds to different guess variables as it does.

Consider a guess value of 1 for variable V. The first equation then has one unknown hf. The second equation then has just one unknown f. The third equation then has just one unknown Re. Finally, the fourth equation will become inconsistent and TK will generate an error term and attempt to iterate on the guess values of V until the inconsistency is resolved.

Unfortunately, here is how TK responds to a guess for V.

Status	Input	Name	Output	Unit	Comment
> Overdefined	.00000114	nu		m^2/s	Kinematic Viscosity
	9.81	g		m/s^2	Acceleration due to gravity
	.00015	ks		m	Equivalent sand roughess height
	1	H		m	Height of fluid in tank
	2	L		m	Length of pipe
> Overdefined	.0158	D		m	Pipe diameter
		hf	2.49516832	m	head loss
		f	.039046341		friction factor
Guess	3.14718884	V		m/s	mean velocity
> Overdefined		Re	.010758309		Reynolds number

And the fourth rule is marked as Inconsistent.

Status	Rule
Satisfied	H+L = V^2/(2*g) + hf
Satisfied	hf = f*L*V^2/(2*g*D)
Satisfied	f = 0.25/(log(ks/(3.7*D) + 5.74/(Re^0.9)))^2
> Inconsistent	Re = D*V/nu

How can an equation involving a guess variable be inconsistent? Did the Iterative Solver give up? Let's follow the steps that TK took to see what happened.

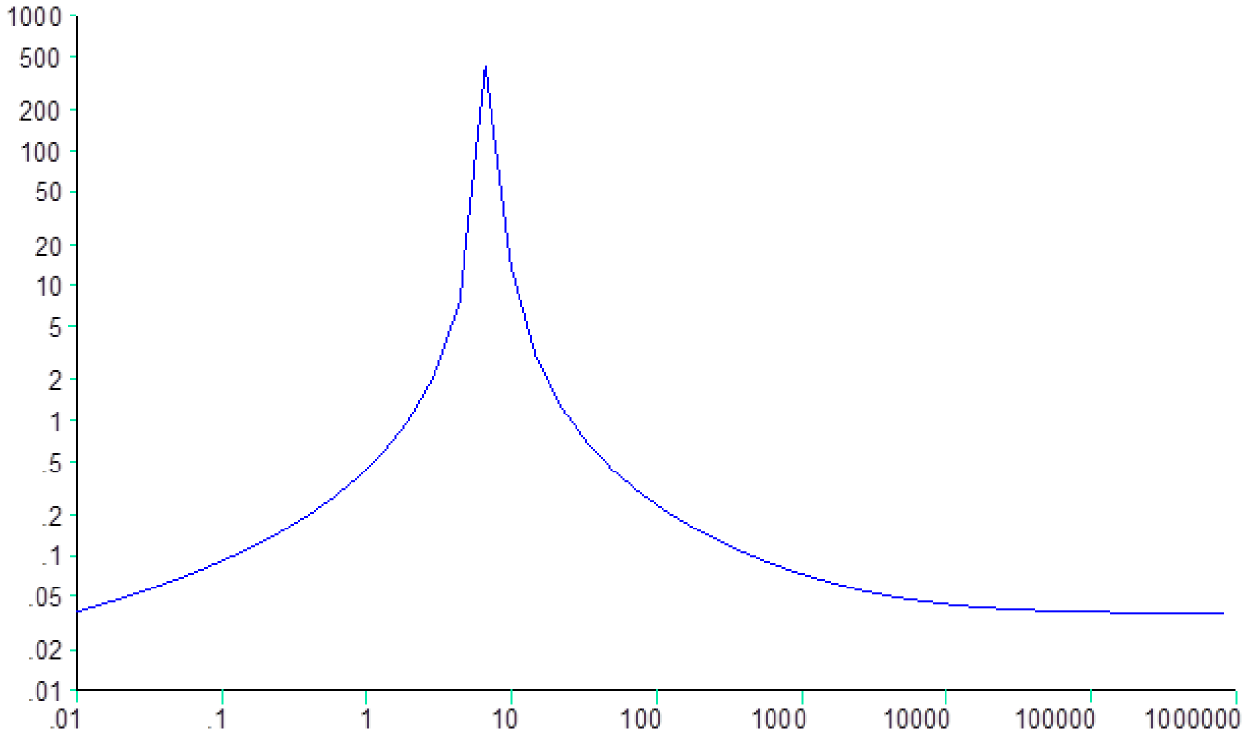
The first equation solved for hf. The second equation solved for f. The third equation solved for Re. The value of Re is then compared against the right side of the fourth equation and no matter values of V TK tries, the value of Re is never equal to D*V/nu.

Here is the inverse equation TK "sees" when solving for Re using the third equation.

$$\left[\frac{5.74}{10^{\left[\frac{-\sqrt{\frac{1}{f}}}{2}\right]}-\frac{ks}{3.7\cdot D}}\right]^{\left[\frac{1}{0.9}\right]}=Re$$

We can see that for a given value of f, TK can compute Re.

Meanwhile, here is a plot of f vs. Re based on the third equation.



We can see from this plot that for a given value of f (on the y-axis), there are two possible values of Re but the inverse equation above only solves for one of the two values of Re. In fact, it will always result in the smaller of the two possible f values. Unfortunately, the fourth equation produces values out by the second solution and TK reports the inconsistency.

Can anything be done to get around this? Of course!

Instead of guessing values for V, use a guess for Re. Look again at the equations to see the sequence of steps TK will take.

$$H + L = \frac{V^2}{2 \cdot g} + hf$$

$$hf = \frac{f \cdot L \cdot V^2}{2 \cdot g \cdot D}$$

$$f = \frac{0.25}{\left[\log\left[\frac{ks}{3.7 \cdot D} + \frac{5.74}{Re^{0.9}}\right]\right]^2}$$

$$Re = \frac{D \cdot V}{nu}$$

The first and second equations are initially skipped because they have two or more unknowns. The third equation solves for f. Looking at the plot above, we know that there is just one f value for each Re value.

The value of V is then computed from the fourth equation and TK makes another pass through the equations. Variable hf is computed from the first equation. The second equation then generates an inconsistency and TK iterates on the value of Re until a solution is found. We know that Re must be greater than 2300 for the flow to be turbulent so we can try a guess of 2301 or 2500.

Status	Input	Name	Output	Unit	Comment
	.00000114	nu		m^2/s	Kinematic Viscosity
	9.81	g		m/s^2	Acceleration due to gravity
	.00015	ks		m	Equivalent sand roughess height
	1	H		m	Height of fluid in tank
	2	L		m	Length of pipe
	.0158	D		m	Pipe diameter
		hf	2.49516832	m	head loss
		f	.039046341		friction factor
		V	3.14718884	m/s	mean velocity
		Re	43618.933		Reynolds number

Success!