March 22, 2005

Case Study: Rapid Conversion of a Technical Paper to a Mathematical Model

Introduction

A technical paper, in PDF form, containing an algorithm with over sixty linear, nonlinear and integral equations is converted to a mathematical model in TK Solver and subsequently linked to Excel and a web application. The math model was completed in one day and the links and refinements to the interface were completed on day two.

The Model Building Process

The technical paper was ESDU Data Item 85007, entitled "Contact Phenomena III: Calculation of Individual Stress Components in Concentrated Elastic Contacts Under Combined Normal and Tangential Loading." It was made available in PDF form.

The PDF file was briefly reviewed and it was noted that two complete examples were provided within the document, either of which could be used to verify a computerized version of the formulas and calculations.

The formulas were entered into TK Solver. Here is a sample of some of the formulas, in TK's MathLook View including the equation numbers from the paper:

$$K = \frac{y1 \cdot theta1 + x1 \cdot theta2 - n}{n^3} \text{; 5.21}$$

$$fbarx1 = beta \cdot \left[2 \cdot (1 - sigma) \cdot z1 \cdot I1 - 2 \cdot sigma \cdot z1 \cdot I3 - \left[\frac{x1 \cdot s1}{G \cdot r1} \right]^2 \cdot L + (1 - 2 \cdot sigma) \cdot \left[K + \frac{z1}{n^2 \cdot r1 \cdot s1} \right] \right] \text{; 5.22}$$

$$fbary1 = beta \cdot \left[2 \cdot (1 - sigma) \cdot z1 \cdot I2 - 2 \cdot sigma \cdot z1 \cdot I3 - \left[\frac{y1 \cdot s1 \cdot r1}{G} \right]^2 \cdot L - (1 - 2 \cdot sigma) \cdot \left[K + \frac{z1 \cdot r1}{n^2 \cdot s1} \right] \right] \text{; 5.23}$$

$$fbarz1 = \frac{-beta \cdot z1^2 \cdot L}{s1^2} \text{; 5.24}$$

$$qbarxy1 = (1 - 2 \cdot sigma) \cdot beta \cdot J - \left[\frac{beta \cdot x1 \cdot y1 \cdot s1^2 \cdot L}{G^2} \right] \text{; 5.25}$$

There are 57 such equations in the math model, in addition to several functions which were added to perform tasks such as generate plots. These will be discussed in more detail later.

The variables in the equations are automatically summarized on TK's variable sheet, including descriptions and units. Here is a portion of the variable sheet, with the inputs grouped at the top.

St	Input	Name	Output	Unit	Comment
					ESDU Item 85007
					Contact Phenomena III: Calculation of Individual Stress
					Components in Concentrated Elastic Contacts
					Under Combined Normal and Tangential Loading
	1	x		mm	x-coordinate of stress point
	0.5	y		mm	y-coordinate of stress point
	0.2	z		mm	z-coordinate of stress point
	1.896	a		mm	Major semi-axis of contact ellipse, a
	0.404	b		mm	Minor semi-axis of contact ellipse, b
	0.3	sigma			Poisson ratio
	0.45	mu			Coefficient of friction for gross sliding
	30	gamma		deg	Angle between force of friction and minor axis of ellipse
	783	p0		MPa	Maximum compressive stress in contact region
		fx	-140.598	MPa	Compressive stress parallel to the x axis
		fy	-247.955	MPa	Compressive stress parallel to the y axis
		fz	-87.987	MPa	Compressive stress parallel to the z axis
		qxy	-17.043	MPa	Shear stress on the xy plane
		qyz	-143.826	MPa	Shear stress on the yz plane
		qzx	-25.752	MPa	Shear stress on the zx plane

The model also includes many auxiliary variables used by the ESDU algorithm. Those variables are grouped together on the variable sheet. During the testing of the model, it was very useful to be able to verify that each intermediate value was computed as specified by the paper. Here is a portion of that listing.

St	Input	Name	Output	Unit	Comment
		n	0.977		Intermediate variable, n = sqrt(1-beta^2)
		b2	0.229		Intermediate variable, b2
		b1	-0.016		Intermediate variable, b1
		b0	0		Intermediate variable, b0
		s1	0.27		Inverse of the upper limit of integration
		I1	0.963		Auxiliary quantity
		I2	4.113		Auxiliary quantity
		I3	5.324		Auxiliary quantity
		r1	3.012		Auxiliary quantity
		G	0.356		Auxiliary quantity
		H	7.309		Auxiliary quantity
		L	2.06		Auxiliary quantity
		N	0.139		Auxiliary quantity
		C	1.212		Auxiliary quantity
		D	0.257		Auxiliary quantity
		E	1.506		Auxiliary quantity
		F	1.122		Auxiliary quantity
		theta1	0.786	rad	Auxiliary quantity
		theta2	0.454	rad	Auxiliary quantity
		J	0.316		Auxiliary quantity
		pdJx1	0.473		Auxiliary quantity
		pdJy1	0.169		Auxiliary quantity
		K	-0.569		Auxiliary quantity

The calculations required three numerical integrations. Fortunately, TK Solver includes built-in functions for those calculations. Here are the integrands.

$$v = u^2 \cdot \left[1 + u^2\right]^{(-1.5)} \cdot \left[1 + beta^2 \cdot u^2\right]^{(-0.5)}$$

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The limits of the integrals were determined from the solution of a sixth order polynomial. It was necessary to find all the roots and then use the largest real root in the upper limit of integration. TK Solver includes a root finder that was perfect for the task, based on Bairstow's algorithm for finding real and complex roots of nth-order polynomials. A loop was added to select the best one for this application. TK's unique capability to mix rule-based equation solving with calls to procedural subroutines was a time-saver.

Here is the procedure that sets up the polynomial, solves it with a call to the Bairstow function and then picks the best solution for the application.

Statement
x = 's1a
x[1] = 1
x[2] = 0
x[3] = 3*b2
x[4] = 0
x[5] = 3*b1
x[6] = 0
x[7] = -3*b0
call Bairstow(x,'s1re,'s1im)
s1 = 0
for i = 1 to 6
if 's1im[i]<>0 then continue
if 's1re[i]<=0 then continue
if 's1re[i]>s1 then s1 = 's1re[i]
next i

The model takes a fraction of a second to solve and the solutions verify the values from the example in the paper.

Interfacing with Excel

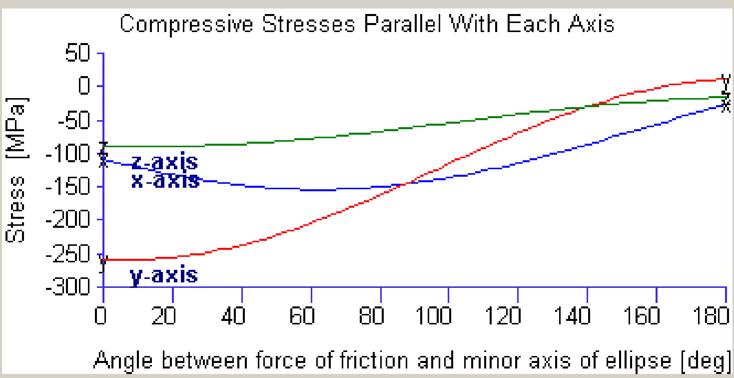
The TK file was linked to an Excel spreadsheet, as shown below. There are several advantages to this. The underlying formulas can be hidden. The interface can be customized. The TK model can be used as a function within a larger spreadsheet application. The spreadsheet and model can be packaged together and shared with anyone.

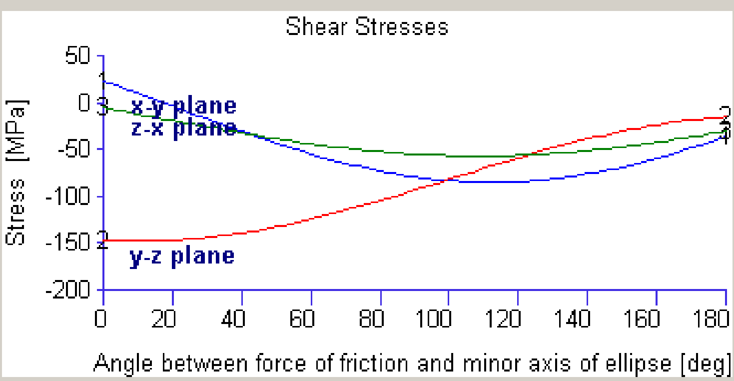
ESDU Item 85007

Contact Phenomena III: Calculation of Individual Stress Components in Concentrated Elastic Contacts Under Combined Normal and Tangential Loading

Inputs	Value	Unit
x-coordinate of stress point	1	mm
y-coordinate of stress point	0.5	mm
z-coordinate of stress point	0.2	mm
Major semi-axis of contact ellipse, a	1.896	mm
Minor semi-axis of contact ellipse, b	0.404	mm
Poisson ratio	0.3	
Coefficient of friction for gross sliding	0.45	
Angle between force of friction and minor axis of ellipse	30	deg
Maximum compressive stress in contact region	783	MPa

Outputs	Value	Unit
Compressive stress parallel to the x axis	-140.598	MPa
Compressive stress parallel to the y axis	-247.9554	MPa
Compressive stress parallel to the z axis	-87.98702	MPa
Shear stress on the xy plane	-17.04325	MPa
Shear stress on the yz plane	-143.8255	MPa
Shear stress on the zx plane	-25.752	MPa

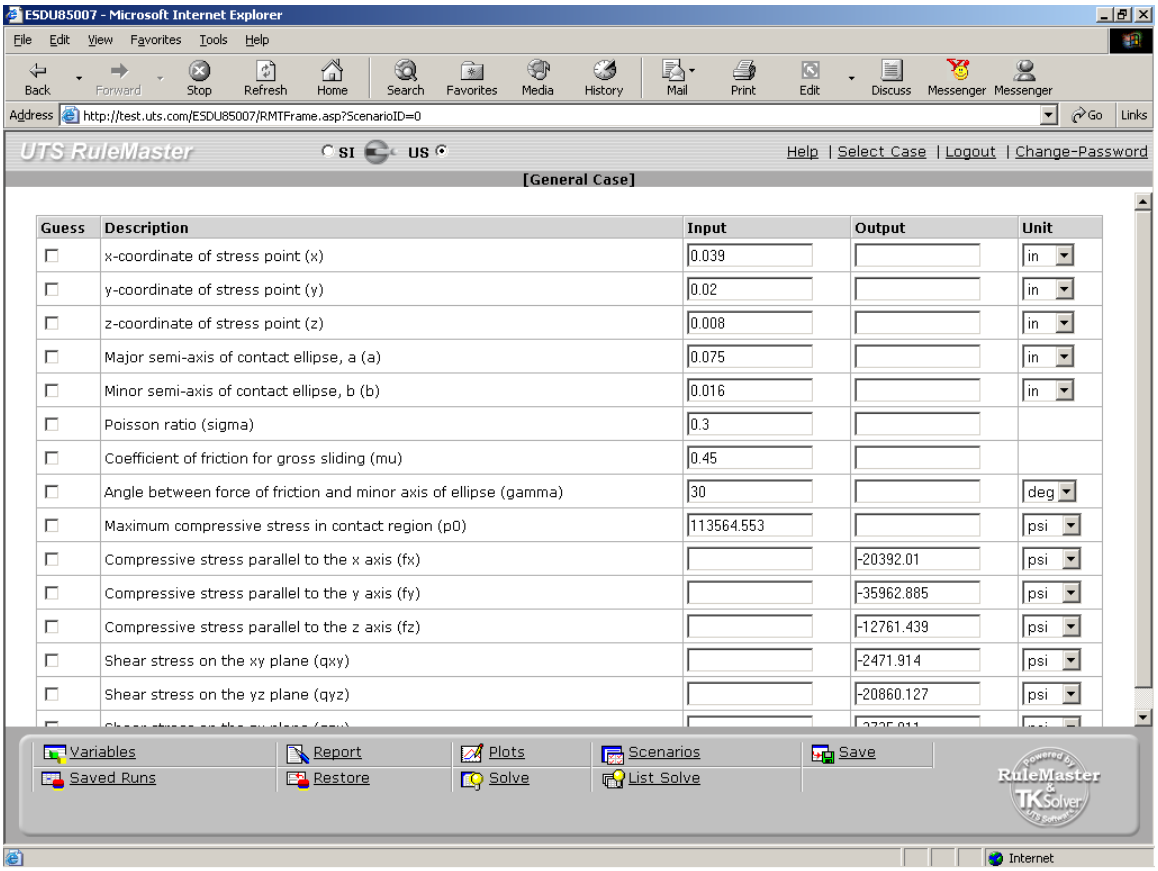




The spreadsheet dynamically updates as the inputs are changed. The plots were added by taking advantage of TK's powerful "List Solve" feature to repeatedly solve the equations over a range of inputs for one of the variables.

Interfacing with the Web

The TK Solver model was also processed using a UTS tool called Galaxy, which automatically builds a browser interface for a web-based application. Again, the variables displayed in the browser interface are dynamically linked to the model.



Odds and Ends

There are a number of additional features of TK Solver that make it an excellent tool for such applications.

Units management is one TK's strengths. In this application, unit conversions are available for all the inputs and outputs. You can globally swap between US and SI units or mix and match as you like. This feature carries through to the Excel and Web interfaces as well.

Table and plot generation is another asset. Any mathematical model built in TK Solver can be enhanced without any additional programming required. As shown, the effects of varying the friction angle are plotted and automatically updated as other variables change. Even the locations of the plot annotations dynamically change.

Backsolving is perhaps TK Solver's most valuable feature. Backsolving allows users to input a desired value for one or more outputs and solve for the required inputs. One example for this model would be to input a stress value and have TK determine the required friction angle or contact dimension. Here is one such example. Note that fy, the component stress with the maximum absolute value, has been input and the contact length, a, has been solved.

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	1	x		mm	x-coordinate of stress point
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		a	1.536	mm	Major semi-axis of contact ellipse, a
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	0.3	sigma			Poisson ratio
	0.45	mu			Coefficient of friction for gross sliding
	30	gamma		deg	Angle between force of friction and minor axis of ellipse
	783	p0		MPa	Maximum compressive stress in contact region
		fx	-121.3	MPa	Compressive stress parallel to the x axis
	-200	fy		MPa	Compressive stress parallel to the y axis
		fz	-62.982	MPa	Compressive stress parallel to the z axis
		qxy	-6.882	MPa	Shear stress on the xy plane
		qyz	-110.772	MPa	Shear stress on the yz plane
		qzx	-22.157	MPa	Shear stress on the zx plane

In this way, a single TK Solver model actually solves much more than the initial intent of the technical paper, with no additional programming required.

Summary

In summary, a technical paper containing dozens of pages of equations is now accompanied by a mathematical model that the reader can immediately use to solve any related problems. With much more functionality than FORTRAN or BASIC translations of the formulas could provide, the TK Solver model gives the reader a dynamic way of turning a complex document into a handful of variables that are easily understand and manipulated. Best of all, the entire process only required a couple days.